Indian Statistical Institute, Bangalore Centre.
Mid-Semester Exam : Graph Theory

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Max. points : 20. **Time Limit : 2 hours.**

Answer any two questions only.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class or assignments, mention it clearly. See the end of the question paper for notations.

1. (a) If G is a simple graph, show that $diam(G) \geq 3 \Rightarrow diam(\bar{G}) \leq 3$. (**5**)

   (b) Show that $2\alpha'(G) = \min_{S \subset V}\{|V(G)| - d(S)\}$ where $d(S) = o(G - S) - |S|$. (**5**)

2. Consider the following algorithm to generate a Minimal spanning tree on a weighted graph G with weight function w. (**10**)

   Step 1 : Initialize $T = G$.

   Step 2 : Let $e = argmax\{w(e') : e' \in T\}$ (As always break ties arbitrarily).

   Step 3 : If e is in a cycle in T, then update $T \leftarrow T - e$ i.e., delete e from T if it belongs to a cycle.

   Step 4 : If T is not a tree, go to Step 2 else go to Step 5.

   Step 5 : Output $M = T$.

   Show that M is a minimal spanning tree.

3. Characterize the graphs G for which the following statements hold. Justify your answers. (**10 = 4 * 2.5**)

   (1) (max. independent set) $\alpha(G) = 1$.
   (2) (max. size of matching) $\alpha'(G) = 1$.
   (3) (min. vertex cover) $\beta(G) = 1$.

(4) (min. edge cover) $\beta'(G) = 1$.

NOTE : In each of the above, you are required to prove a statement of the form $\ldots(G) = 1$ *iff* G *is* $\ldots$.

Some notations :

- G is assumed to be a finite simple graph everywhere.
- If G is a graph, $\bar{G}$ (the complement) is defined as the graph with vertex set $V(G)$ and edge set $\{(u, v) : (u, v) \notin E(G)\}$.
- d_G is defined as the usual graph metric when all edge weights are taken to be 1.
- $o(G)$ - Number of odd components in a graph G.
- $\alpha'(G)$ - Maximum independent edge set ; $\beta'(G)$ - Minimum edge cover.
- $\alpha(G)$ - Maximum independent vertex set ; $\beta(G)$ - Minimum vertex cover.